Algebra 2 Made Easy Handbook

Next Generation Learning Standards Edition

By:

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Dedication

This book is dedicated to our mother, MaryAnn Casey. She was a devoted teacher, a lifelong learner, and someone who never stopped believing in her students' ability to succeed. Even after she retired, her desire to help students learn remained strong, and writing these books allowed her to keep teaching beyond the classroom.

As both her daughter and a fellow math teacher, I was grateful for the opportunities to share in this work with her over the years. Finishing this book in her memory has been a privilege. May her love of mathematics and her commitment to helping others continue to guide all who open these pages.

Debra Rainha – Math Teacher at Andover (MA) High School
Cathleen Krom
Eileen Casey

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Next Generation Learning Standards Edition.

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Introduction

This is a quick reference guide, or “how to do it book”. Implementing new teaching techniques and providing deeper understanding of the content of the course is the job of the classroom teacher. Some topics I have included may go beyond the NGLS and will help with working with the NGLS. Always follow the directions given by your classroom teacher.

As the curriculum for *Algebra II, Next Generation Learning Standards* are implemented, it is my hope that this student friendly book will assist students in becoming “college and career ready”, supporting the main goals of our educational process.

Sincerely,

MaryAnn Casey,
B.S. Mathematics, M.S. Education

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1.8

SOLVING RATIONAL EQUATIONS & INEQUALITIES

SOLVING RATIONAL EQUATIONS

Rational Equations: Equations that contain algebraic fractions (rational expressions). There are several methods available to solve them.

Proportions: Two rational expressions are equal to each other. "Cross multiply" is the common name for solving these.

Steps:

1) Multiply the numerator of each fraction by the denominator of the other.
2) Solve the resulting equation for the variable.
3) Checking is essential.

Note: Remember that some answers may cause a denominator in the problem to be equal to zero and must, therefore, be rejected. This will be evident when you check your answer(s) in the original fractions.

Examples

❶ $\dfrac{x+4}{3} = \dfrac{4}{x}$

$x(x+4) = 12$

$x^2 + 4x - 12 = 0$

$(x+6)(x-2) = 0$

$x + 6 = 0 \quad x - 2 = 0$

$x = -6, \quad x = 2$

A zero denominator is not created if $x = -6$ or if $x = 2$. Both are good answers. Check both in the original equation.

Check: $\dfrac{x+4}{3} = \dfrac{4}{x}$

$\dfrac{-6+4}{3} = \dfrac{4}{-6} \qquad \dfrac{2+4}{3} = \dfrac{4}{2}$

$\dfrac{-2}{3} = \dfrac{2}{-3} \qquad \dfrac{6}{3} = \dfrac{4}{2}$

$-\dfrac{2}{3} = -\dfrac{2}{3}$ ✓ $\qquad 2 = 2$ ✓

Both –6 and 2 check in this problem.

❷ If an equation is not a proportion, you can make it into one by changing each side of the equation into a single fraction, each side with its own LCD. Then cross multiply and solve.

$$\frac{1}{9} + \frac{1}{2y} = \frac{1}{y^2}$$

$$\frac{2y+9}{18y} = \frac{1}{y^2}$$

$$(y^2)(2y+9) = (1)(18y)$$

$$2y^3 + 9y^2 = 18y$$

$$2y^3 + 9y^2 - 18y = 0$$

$$y(2y^2 + 9y - 18) = 0$$

$$y(y+6)(2y-3) = 0$$

$y = 0$ $\quad y + 6 = 0$ $\quad 2y - 3 = 0$

$y = 0$ $\quad y = -6$ $\quad y = \frac{3}{2}$

$y = 0$ has to be rejected. The answers are –6 and $\frac{3}{2}$

Check: $\frac{1}{9} + \frac{1}{2y} = \frac{1}{y^2}$

$$\frac{1}{9} + \frac{1}{2(0)} = \frac{1}{0^2}$$

Division by 0 is undefined. 0 is an extraneous root.

$$\frac{1}{9} + \frac{1}{2(-6)} = \frac{1}{(-6)^2}$$

$$\frac{1}{9} - \frac{1}{12} = \frac{1}{36}$$

$$\frac{4}{36} - \frac{3}{36} = \frac{1}{36}$$

$\frac{1}{36} = \frac{1}{36}$ ***checks*** ✓

Since $\frac{3}{2} = 1.5$, using 1.5 for the check is easier.

$$\frac{1}{9} + \frac{1}{2(1.5)} = \frac{1}{1.5^2}$$

$$\frac{1}{9} + \frac{1}{3} = \frac{1}{2.25}$$

$$\frac{1+3}{9} = \frac{1}{2.25}$$

$$\frac{4}{9} = \frac{1}{2.25}$$

$.444... = .444...$ ***checks*** ✓

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Solve by Clearing the Fractions: This method removes the fractions from the problem. (*Note*: There is some controversy about solving rational equations. Follow the method your teacher prefers.)

Steps:

1) Find the Least Common Multiple (LCM) of ALL the denominators in the equation. Keep it in factor form.
2) Using the distributive property, multiply each term in the equation by the LCM, canceling whenever possible. This process removes the fractions completely.
3) Solve.
4) Check. Watch for zero denominators. All solutions may not check.

Examples

❶

$$\frac{1}{9}+\frac{1}{2x}=\frac{1}{x^2}$$

$$\left(\frac{1}{9}+\frac{1}{2x}=\frac{1}{x^2}\right)\left(18x^2\right)$$

$$2x^2+9x=18$$

$$2x^2+9x-18=0$$

$$(2x-3)(x+6)=0$$

$$2x-3=0 \qquad x+6=0$$

$$x=\frac{3}{2} \qquad x=-6$$

Check:

$$\frac{1}{9}+\frac{1}{2(-6)}=\frac{1}{(-6)^2} \qquad \frac{1}{9}+\frac{1}{2\left(\frac{3}{2}\right)}=\frac{1}{\left(\frac{3}{2}\right)^2}$$

$$\frac{1}{9}+\frac{1}{12}=\frac{1}{36} \qquad \frac{1}{9}+\frac{1}{3}=\frac{1}{\frac{9}{4}}$$

$$\frac{12}{108}-\frac{9}{108}=\frac{1}{36} \qquad \frac{1}{9}+\frac{3}{9}=\frac{4}{9}$$

$$\frac{3}{108}=\frac{1}{36} \qquad \frac{4}{9}=\frac{4}{9}$$

$$\frac{1}{36}=\frac{1}{36}$$

$$SS=\left\{-6,\ \frac{3}{2}\right\}$$

❷ $$\frac{3}{x}+\frac{2}{x+2}=\frac{-x}{x+2}$$

$$\frac{3}{x}+\frac{2}{x+2}+\frac{x}{x+2}=0$$ move everything to the left side.

$$x(x+2)\left(\frac{3}{x}+\frac{x+2}{x+2}=0\right)$$ multiply by $x(x+2)$

$$3(x+2)+x(x+2)=0$$

$$3x+6+x^2+2x=0$$

$$x^2+5x+6=0$$

$$(x+3)(x+2)=0$$

$$x+3=0 \qquad x+2=0$$

$$x=-3 \qquad x=-2$$

Check: $$\frac{3}{x}+\frac{2}{x+2}=\frac{-x}{x+2}$$

$$\frac{3}{-3}+\frac{2}{-3+2}=\frac{-(-3)}{-3+2}$$

$$-1+-2=\frac{3}{-1}$$

$$-3=-3$$ *checks* ✓

$$\frac{3}{-2}+\frac{2}{-2+2}=\frac{-(-2)}{-2+2}$$ Division by 0 is undefined.

$$-1+\frac{-2}{0}=\frac{2}{0}$$

-2 is an extraneous root.

$$SS=\{-3\}$$

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SOLVING RATIONAL INEQUALITIES

A **rational inequality** includes an inequality symbol instead of an equal sign.

Steps:

1) Determine what values of the variable will cause a fraction to be undefined.
2) Change the inequality symbol to = and solve the equation as usual.
3) On a number line, graph the solution(s) to the equation *and* the values of the variable found in step 1. These are called critical points.
4) The number line will be divided into sections. Test a number in each section of the number line to see which section(s) of the number line contain the numbers to make the inequality true.
5) Write the solution in inequality or set builder form.

Examples

❶ $\frac{3x-4}{8} < \frac{4x-3}{4}$

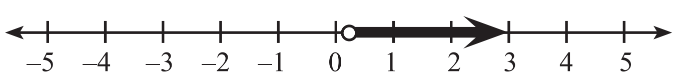
$4(3x - 4) = 8(4x - 3)$

$12x - 16 = 8(4x - 3)$

$12x - 16 = 32x - 24$

$-20x = -8$

$x = \frac{-8}{-20} = \frac{2}{5}$



Since this inequality has no variables in the denominator, only one point can be graphed. Test a value from each part of the number line in the original problem.

Test : 0 $\frac{3(0)-4}{8} < \frac{4(0)-3}{4}$

$-\frac{1}{2} < -\frac{3}{4}$ *False*

Test : 5 $\frac{3(5)-4}{8} < \frac{4(5)-3}{4}$

$\frac{11}{8} < \frac{17}{4}$ *True*

Solution: $\{x : x > \frac{2}{5}\}$

❷ $\frac{x}{3} - \frac{x+1}{4} < \frac{1}{x-2}$, $x = 2$ is excluded., $x \neq 2$

LCM is $12(x-2)$.Clear fractions, (see page 45) make into an equation.

$$12(x-2)\left(\frac{x}{3} - \frac{x+1}{4} = \frac{1}{x-2}\right)$$

$$4x(x-2) - 3(x+1)(x-2) = 12$$

$$4x^2 - 8x - 3\left(x^2 - x - 2\right) = 12$$

$$4x^2 - 8x - 3x^2 + 3x + 6 = 12$$

$$x^2 - 5x - 6 = 0$$

$$(x-6)(x+1) = 0 \checkmark$$

$$x = 6 \qquad x = -1$$

Critical Points are –1, 2, 6. Open circles on all 3 points. Test –5, 0, 3, and 10 because the number line has 4 sections on this example.

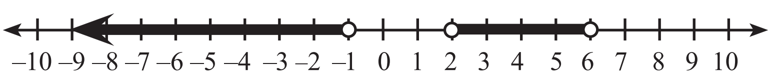
$$\frac{x}{3} - \frac{x+1}{4} < \frac{1}{x-2}$$

Test -5: $\frac{-5}{3} - \frac{-5+1}{4} < \frac{1}{-5-2}$ ***True***

Test 0 : $\frac{0}{3} - \frac{0+1}{4} < \frac{1}{0-2}$ ***False***

Test 3 : $\frac{3}{3} - \frac{3+1}{4} < \frac{1}{3-2}$ ***True***

Test 10 : $\frac{10}{3} - \frac{10+1}{4} < \frac{1}{10-2}$ ***False***



Solution: $(x < -1)$ *or* $(2 < x < 6)$ *or* $\{x|(x < -1)$ *or* $(2 < x < 6)\}$

❸ $\frac{7}{2x} - \frac{2}{x} \geq \frac{3}{2}$ $\quad x \neq 0$

$$\left(\frac{7}{2x} - \frac{2}{x} = \frac{3}{2}\right)(2x)$$

$$7 - 4 = 3x$$

$$3 = 3x$$

$$x = 1$$

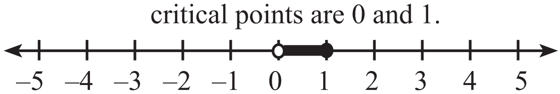
Test (-1) : $\frac{7}{-2} - \frac{2}{-1} \geq \frac{3}{2}$; $-3\frac{1}{2} + 2 \geq \frac{3}{2}$; ***False***

Test $\left(\frac{1}{2}\right)$: $\frac{7}{1} - \frac{2}{\frac{1}{2}} \geq \frac{3}{2}$; $7 - 4 \geq \frac{3}{2}$; ***True***

Test (2) : $\frac{7}{4} - \frac{2}{2} \geq \frac{3}{2}$; $\frac{7}{4} - 1 \geq \frac{3}{2}$; ***False***

Solution : $\{x \mid 0 < x \leq 1\}$

critical points are 0 and 1.



Open circle on 0 and solid circle on 1

1.8

SOLVING INEQUALITIES BY GRAPHING

To solve an inequality graphically, make a graph of the problem as if it were an equation. Then choose a point on either side of the line or curve and test the (x, y) coordinates of that point in the original inequality. Do not pick a point on the graphed line to test.

If the x and y values of the point chosen to test make the inequality true, shade the graph on the side to include that point. That is the solution. If it is false, the opposite side is the solution and is to be shaded.

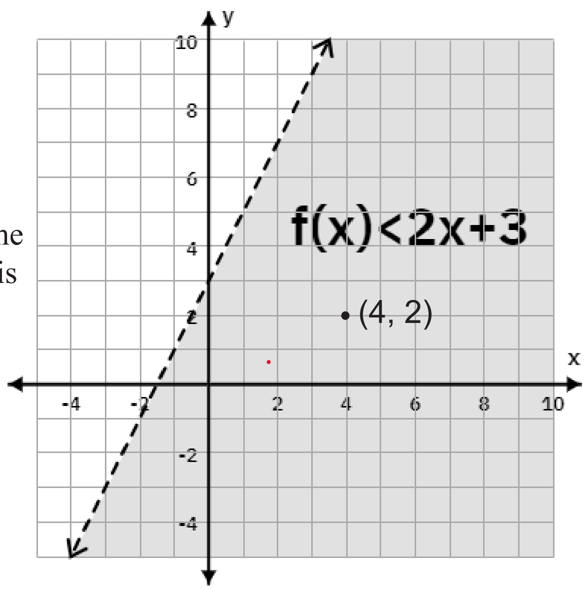
Remember to make a broken graph line if the inequality is exclusive and contains just < or >. If it is $\leq$ or $\geq$, it is inclusive, the graphed line is solid. Do not pick a point on the graphed line to test.

Examples

❶ Solve by graphing. $F(x) < 2x + 3$. Note that this is exclusive, so the graph line will be broken.

Test (4, 2)
$2 < 2(4) + 3$
$2 < 11$ *True*

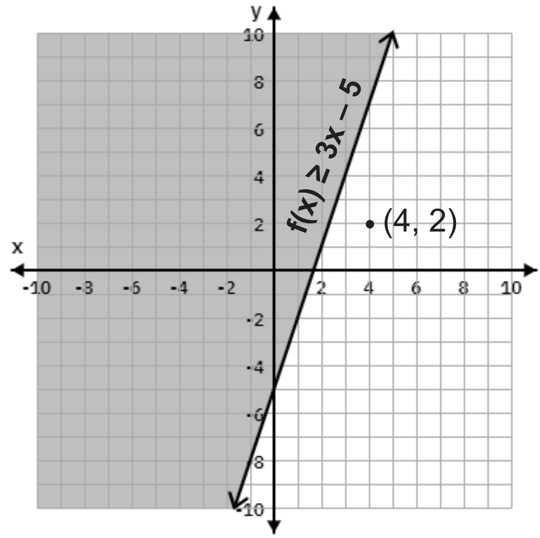
Shade the same side as (4, 2) is located on.



❷ Solve by graphing. $F(x) \geq 3x - 5$. This is an inclusive inequality. The line will be solid.

Test (4, 2)
$2 \geq 3(4) - 5$
$2 \geq 12 - 5$
$2 \geq 7$ *False*

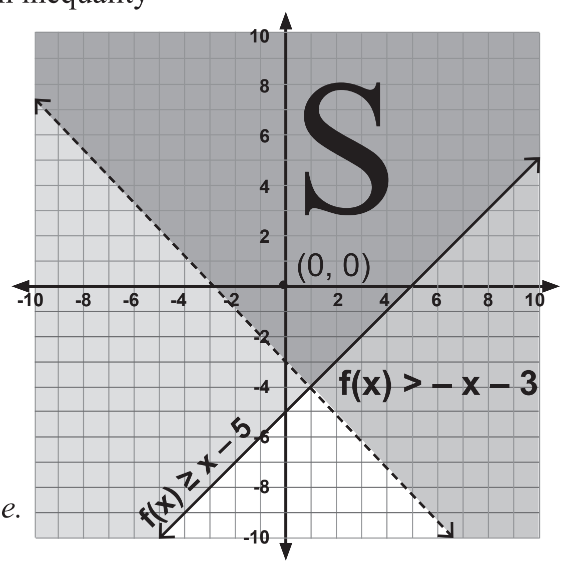
Shade the side opposite (4, 2).



To solve a system of inequalities by graphing, follow the procedure described above for both graphs but put them on the same grid. If the solutions overlap, that is the solution set for the system.

❸ Solve by graphing:
Test (0, 0) in each inequality
$f(x) \geq x - 5$
$0 \geq 0 - 5$
$0 \geq -5$ *True*
Shade the side with (0, 0).
Solid line.
Label.

$f(x) > - x - 3$
$0 > 0 - 3$
$0 > -3$ *True*
Shade the same side as (0, 0).
Broken graph line.
Label.



2.7 LOGARITHMIC FUNCTIONS

LOGARITHMS – ALSO KNOWN AS LOGS

Logarithm: The inverse of an exponential function. To find the inverse of a function, we exchange the x and y and then solve for y. In an exponential function, the exchange of x and y will cause y to be an exponent. In order to solve for y, we use logarithms. The form of a log function is $y = \log_b x$ where b is the base, is positive and is not equal to one. This is read, "y is log to the base b of x." (Review for Finding Inverses see page 124.)

When writing an exponential equation in log form, the parts of the exponential equation have specific positions in the equivalent log equation.

Exponential Form: $x = b^y$ **Log form:** $\log_b x = y$

$\boxed{\log_b x = y}$ ← **exponent**

base **argument**

Example Find the inverse of each of the following functions.

$$f(x) = 12^x$$
$$y = 12^x$$
$$f^{-1}:\ x = 12^y$$
$$y = \log_{12} x$$
$$f^{-1}(x) = \log_{12} x$$

$$f(x) = 5^{2x}$$
$$y = 5^{2x}$$
$$f^{-1}:\ x = 5^{2y}$$
$$2y = \log_5 x$$
$$\boldsymbol{y = \frac{\log_5 x}{2}} \text{ or } \boldsymbol{y = \frac{1}{2}\log_5 x}$$
$$\boldsymbol{f^{-1}(x) = \frac{\log_5 x}{2}} \text{ or } \boldsymbol{f^{-1}(x) = \frac{1}{2}\log_5 x}$$

Note: The example above require finding the inverse of an exponential function. Once the inverse is developed by exchanging x and y, change the new equation into a logarithm to solve for y. That is the inverse function.

Domain and Range: Since log functions are inverses of exponential functions, the domain of the log function is all the positive real numbers. The range is all real numbers.

Evaluate a Log: This means to find the exponent that must be applied to the base to equal the given number called the argument.

Examples 1, 2, and 3 contain familiar numbers. Ex. 4 requires a calculator.

❶ Evaluate $\log_2 8$.

2 is the base. What exponent must be applied to 2 to make it equal to 8? Since $2^3 = 8$, the answer is $\log_2 8 = 3$.

❷ Evaluate $\log_9 3$.

9 is the base. What exponent can be applied to 9 to make it equal to 3? Since $9^{\frac{1}{2}} = 3$, the answer is $\log_9 3 = \frac{1}{2}$.

❸ Evaluate $\log_4 \frac{1}{16}$.

4 is the base. $\frac{1}{4^2} = \frac{1}{16}$

$4^{-2} = \frac{1}{16}$

The solution is $\log_4 \frac{1}{16} = -2$.

❹ Evaluate $\log_5 2$

The base is 5 and, when raised to a power it must equal 2. This is not an answer that is familiar. Use the calculator. Change to base ten first if your calculator only uses common logs by dividing log 2 by log 5.

$\log_5 2$

$\frac{\log 2}{\log 5}$

$\log_5 2 \approx .4307$

<u>Properties and Laws of Logarithms (log)</u>:

The properties and rules of exponents (See page 2.) also apply to logs. They are applied to various types of log problems in order to solve for a variable.

- Product: Add the logs. $\log_b(\boldsymbol{mn}) = \log_b \boldsymbol{m} + \log_b \boldsymbol{n}$
- Quotient: Subtract the logs. $\log_{\boldsymbol{b}}\left(\frac{\boldsymbol{m}}{\boldsymbol{n}}\right) = \log_{\boldsymbol{b}} \boldsymbol{m} - \log_{\boldsymbol{b}} \boldsymbol{n}$
- Power to a Power: Multiply. Since "*n*" is already an exponent, multiply the log (also an exponent) by "*n*". $\log_b(\boldsymbol{m^n}) = \boldsymbol{n} \bullet \log_b \boldsymbol{m}$

Depending on the type of problem, we may need to change a single log to an expanded form, or to change an expanded log back into the form of a single log. To combine into a single log, the bases of the expanded log expression must be equal. (See Common Logs page 111.)

(*See examples on the next page*.)

Examples Expand.

❶ $\log_5(3x) = \log_5 3 + \log_5 x$

❷ $\log_3(7xy^2) = \log_3 7 + \log_3 x + 2\log_3 y$

❸ $\log\left(\dfrac{2x}{5y}\right) = \log 2 + \log x - (\log 5 + \log y)$

If no base is shown, the base is 10 and the log is called a common log.

Examples Combine into a single log.

❶ $\log_3 7 + \log_3 12 = \log_3(7 \bullet 12) = \log_3(84)$

❷ $\log x + \log(x+2) = \log[x(x+2)]$

❸ $\log x + 5\log y - 2\log z = \log\left(\dfrac{xy^5}{z^2}\right)$

<u>Logs and Radicals</u>: If a single log is shown in radical form, change the radical to a fractional exponent before expanding the log.

Examples Expand each logarithm.

❶ $\log\sqrt{x+3} = \log(x+3)^{\frac{1}{2}} = \dfrac{1}{2}\log(x+3)$

❷ $\log\sqrt[2]{xy^3} = \log(xy^3)^{\frac{1}{2}} = \dfrac{1}{2}(\log x + 3\log y)$ ***or*** $\dfrac{1}{2}\log x + \dfrac{3}{2}\log y$

2.7

Write in Log Form: Exponential equations can be put in log form, or log equations can be written in exponential form. Either form can be used when solving equations.

Examples Write in log form.

❶ $7^x = 80$; $\log_7 80 = x$

❷ $2^x = 3$; $\log_2 3 = x$

Examples Write in exponential form.

❶ $\log_6 x = 5$; $6^5 = x$

❷ $\log_5 90 = x$; $5^x = 90$

Common Logs -- Logs Using Base 10

Our decimal number system is base 10. Since that is the most common set of numbers that we deal with, the logs of those numbers are called common logs. We can work with common logs in our calculators. The absence of a base in a log expression or equation lets us know automatically that the base is 10. If another base is indicated, we can change the expression into a common log so the calculator can be used.

$\log x = 3$ means the same as $\log_{10} x = 3$. In exponential form it is $10^3 = x$ which can then be solved. $x = 1000$

$\log x = 2.57863921$ means $\log_{10} x = 2.57863921$. In exponential form this is $10^{2.57863921} = x$. This can be solved using a calculator. $x = 379$

Change of Base Formula

Logarithms in bases other than 10 can be translated to an equivalent common log using the formula: $\log_{\boldsymbol{b}} \boldsymbol{x} = \dfrac{\log \boldsymbol{x}}{\log \boldsymbol{b}}$. A problem with mixed log bases in it should be changed to all common logs before solving.

Examples Change the following to common logs.

❶ $\log_5 2 = \dfrac{\log 2}{\log 5} = 0.4307$

❷ $\log_2 \boldsymbol{x} + \log_3 \boldsymbol{x} = \dfrac{\log \boldsymbol{x}}{\log 2} + \dfrac{\log \boldsymbol{x}}{\log 3}$

(*Examples of Common Logs on next page*)

Examples of Common Logs

<table>
<tr>
<td>$\log N = 2.798325$
$10^{2.798325} = N$
$N = 628.528536$</td>
<td rowspan="2">Evaluate log 15
Use log button, type in 15, Enter.
$\log 15 = 1.176091259$
1.176091259
Check
$\log 15 = x$
Means: $10^{x} = 15$
$10^{1.176091259} = 15$</td>
<td rowspan="2">Solve & round to 4 places.
$5^{2x+4} = 32$
$\log(5^{2x+4}) = \log(32)$
$(2x+4)\log 5 = \log 32$
$2x+4 = \frac{\log 32}{\log 5}$
$x = \left(\left(\frac{(\log 32)}{(\log 5)}\right) - 4\right) \div 2$
$x = -0.9233086048$
$x \approx -0.9233$
Check
$5^{(2(-0.9233086048)+4)} = 32$
Be careful with the parentheses!</td>
</tr>
<tr>
<td>$\log 10 = x$
$10^{x} = 10$
$x = 1$</td>
</tr>
</table>

Solving Equations Using Logs: Use the laws of logarithms or the relationship between logarithmic and exponential form of an expression to solve log equations. Any logarithmic equation can be changed into a common log equation by changing the logarithmic equation into exponential form first or by using the change of base formula.

Note: Solving equations containing exponents and the use of logs to solve equations are closely related. Analyze each problem to find the most efficient method. Exponential equations involving *e* will require the use of the natural log function – abbreviated "ln" on the calculator. All properties and laws of logs in general are applied to ln.

What to do if given a value for log *x*: When the solution says "log x = a number", the number is the exponent of 10 that is required to find x. Use your calculator to find x.

Example $\log x = 2.4759$

$10^{2.4759} = x$

$x = 299.1575722$

2.7

The following examples show the use of logs in various types of equations and are rounded to a convenient place value. Be sure and follow directions about rounding when taking a test.

Examples

❶ $8^{x} = 50$

$\log 8^{x} = \log 50$

$x \log 8 = \log 50$

$x = \dfrac{\log 50}{\log 8}$

$x \approx 1.8813$

❷ $x^7 = 497$

$\log x^7 = \log 497$

$7 \log x = \log 497$

$\log x = \dfrac{\log 497}{7}$

$\log x = .385193$

$10^{.385193} = x$

$x \approx 2.43$

❸ $x^e = 40$

$\ln x^e = \ln 40$

$e \ln x = \ln 40$

$\ln x = \dfrac{\ln 40}{e}$

$\ln x = 1.35706$

$x = e^{1.35706}$ (e is on the calculator.)

$x \approx 3.8848$

Examples 2 and 3 can be done using the reciprocal exponent. However, logs are used here to demonstrate solving with them.

❹ $\log_3(x-3) + \log_3(x+5) = 2$

Write as a single log: $\log_3\left[(x-3)(x+5)\right] = 2$

Change to exponent form: $(x-3)(x+5) = 3^2$

Solve : $x^2 + 2x - 15 = 9$

$x^2 + 2x - 24 = 0$

$(x-4)(x+6) = 0$

$x = 4$ and $x = -6$ *reject*

Remember the arguments, $(x - 3)$ and $(x + 5)$, must be > 0.

Exponential Growth and Decay and Logs: Although some exponential growth or decay problems can be solved without logs, it is often easier to use logs – and sometimes necessary.

Remember the formulas: $A_f = A_0(1+r)^t$ and $A_f = A_0(1-r)^t$ (See page 130.)

Examples

❶ Emily deposited $5000 in an account at 4% interest. She now has $6300. How many years was the money in the account? Round to the nearest 10th.

$A_f = 6300 \qquad A_0 = 5000 \qquad r = 4\% \qquad t = ?$

$A_f = A_0(1+r)^t$

$6300 = 5000(1+0.04)^t$

$\log 6300 = \log 5000 + t\log(1.04)$

$\log 6300 - \log 5000 = t\log(1.04)$

$\dfrac{\log 6300 - \log 5000}{\log(1.04)} = t$

$t = 5.892; \quad t \approx 5.9$ ***years***

Alternate Solution

$6300 = 5000(1+0.04)^t$

$\dfrac{6300}{5000} = (1.04)^t$

$1.26 = 1.04^t$

$\log 1.26 = \log 1.04^t$

$\log 1.26 = t\log 1.04$

$\dfrac{\log 1.26}{\log 1.04} = t$

$t = 5.892; \quad t \approx 5.9$ ***years***

❷ Find the rate needed to increase $200 to $425 in 7 years.

$A_f = 425 \qquad A_0 = 200 \qquad r = ? \qquad t = 7$

$425 = 200(1+r)^7$

$\log 425 = \log 200 + 7\log(1+r)$

$\log 425 - \log 200 = 7\log(1+r)$

$\dfrac{\log 425 - \log 200}{7} = \log(1+r)$

$\log(1+r) = 0.0467655621$

$10^{0.0467655621} = 1+r$

$r = 0.11369$ ***or*** $\approx 11.4\%$

Alternate Solution

$425 = 200(1+r)^7$

$\dfrac{425}{200} = (1+r)^7$

$2.125 = (1+r)^7$

$\log 2.125 = \log(1+r)^7$

$\log 2.125 = 7\log(1+r)$

$\dfrac{\log 2.125}{7} = \log(1+r)$

$0.0467655621 = \log(1+r)$

$10^{0.0467655621} = 1+r$

$1.11369 = 1+r$

$r = 0.11369$ ***or*** $\approx 11.4\%$

3.5 GRAPHING TRIG FUNCTIONS

When graphing, the x value of the point graphed represents the angle itself. It is usually in radian form but can be in degrees. Label accordingly. The angle can be represented by θ, x, or any upper case letter. The range, or y values, are the values of the trig functions. The points to be graphed will have the form $(\theta, \sin\theta)$, $(x, \sin x)$ or $(A, \sin A)$. The domain is $-\infty < \theta < \infty$ unless restricted.

Function	Domain	Range
Cosine	$-\infty < \theta < \infty$	$-1 \le y \le 1$
Sine	$-\infty < \theta < \infty$	$-1 \le y \le 1$
Tangent	$-\infty < \theta < \infty$ except odd multiples of $\frac{\pi}{2}$	$-\infty < y < \infty$

Periodic Functions: The values of the trig functions repeat in a pattern. These are also called cyclical functions. As the terminal side of the angle continues to rotate counter-clockwise around the unit circle, the trig functions repeat themselves, making them cyclical.

Sine and Cosine Graphs: Both basic graphs have one complete cycle (one complete curve) within the domain of $0 \le \theta \le 2\pi$. The horizontal x-axis is the principal axis and is labeled with values of θ in radian form. The vertical axis is the y-axis and represents the value of the function at the appropriate value of θ. The y-axis is labeled with integers and each function has a maximum value of 1 and a minimum value of –1. They each have an amplitude of 1, a frequency of 1, and a period of 2π.

Key Points: Each graph has five key values of θ for which the function is equal to a maximum value (1), minimum value (–1), or zero. The key values of θ for both functions are $0, \frac{\pi}{2}, \pi, \frac{3\pi}{2}, 2\pi$. The key points are located at these five values of θ and additional points in between them are plotted by estimating the y-values on the graph. Use the special angles for the additional points. A trig graph is labeled as shown. (Match the label on the horizontal axis to the variable representing the angle.)

Transformations performed on the basic graphs may change the location of the maximum, minimum, and zero values when the equations are graphed, but there will still be five key points in one cycle of each of the graphs. To determine the appropriate locations, we will find the five key points first.

- Graph of $f(x) = a\cos b(\theta + c) + d$ where $a = 1, b = 1, c = 0, d = 0;\ 0 \le \theta \le 2\pi$

Key Points

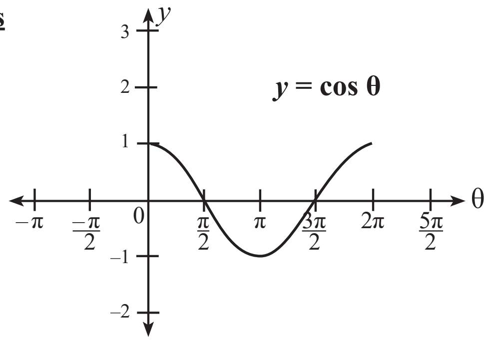
$(0, 1)$

$\left(\frac{\pi}{2}, 0\right)$

$(\pi, -1)$

$\left(\frac{3\pi}{2}, 0\right)$

$(2\pi, 1)$



- Graph of $f(x) = a\sin b(\theta + c) + d$ where $a = 1, b = 1, c = 0, d = 0;\ 0 \le \theta \le 2\pi$

Key Points

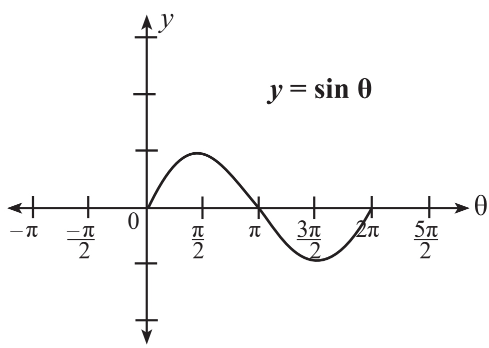
$(0, 0)$

$\left(\frac{\pi}{2}, 1\right)$

$(\pi, 0)$

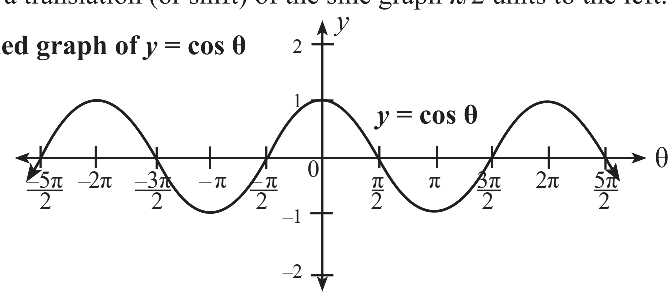
$\left(\frac{3\pi}{2}, -1\right)$

$(2\pi, 0)$



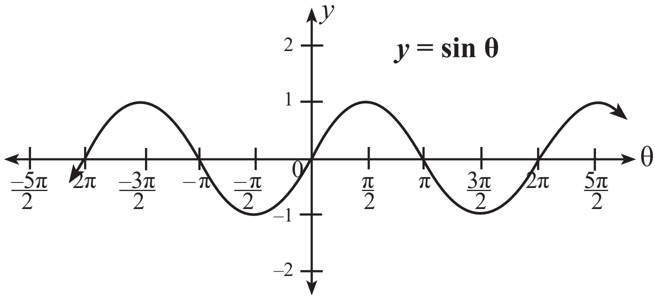
Due to the cyclical nature of trig graphs, both the cosine and sine graphs repeat their curves in both the negative and positive directions. The cosine graph is a translation (or shift) of the sine graph $\pi/2$ units to the left.

Expanded graph of $y = \cos\theta$



Graphs of Trig Functions

Expanded graph of $y = \sin \theta$



AMPLITUDE, FREQUENCY, AND PERIOD

The equations $y = a \sin b(x + c) + d$ or $y = a \cos b(x + c) + d$ provide additional information about trig graphs.

a represents the amplitude, *b* is the frequency, *c* horizontal shift, and *d* is the vertical shift.

In the basic graphs, *a* and *b* are both 1, and *c* and d are both zero. *a* and *b* involve the shape of the graph, the width and vertical measure of the curve. *c* and *d* involve placement of the curve on the axes.

<u>Amplitude</u>: One-half the vertical distance from the minimum point to the maximum point on the graph. $|a|$ is the value of the amplitude.

Example $y = 3 \cos x$. The amplitude is 3. The maximum point of the graph is at $y = 3$, and the minimum point is at $y = -3$.

<u>Frequency</u>: The number of times one complete cycle or pattern of the function occurs within 2π. *b* is the frequency.

Example $y = \cos 2x$. The frequency is 2. There are two complete cosine curves within 2π radians.

<u>Period</u>: The interval in degrees or radians that contains one complete cycle of the function. The period is found by dividing 2π by *b*.

Example $y = \cos 2x$. *b* is 2 so the period is $\frac{2\pi}{2}$ or π. This means that one complete cycle of the curve occurs in π radians.

Sketching the Graphs

Examples

❶ $y = 3\cos\theta$; $0 \le \theta \le 2\pi$

Amplitude: $a = 3$ Highest point is 3, lowest is –3.

Frequency: $b = 1$ One complete cycle occurs once in 2π.

Period: $\frac{2\pi}{1} = 2\pi$ Only one complete cycle appears in the interval 2π.

Key Points

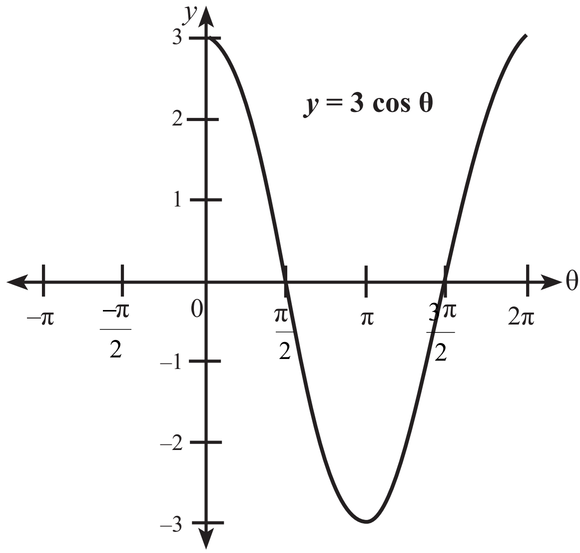
$(0, 3)$

$(\frac{\pi}{2}, 0)$

$(\pi, -3)$

$(\frac{3\pi}{2}, 0)$

$(2\pi, 3)$



❷ $y = \sin(2\theta)$; $-\pi \le \theta \le 2\pi$ (Note that the domain is $-\pi$ to 2π)

Amplitude: $a = 1$: Maximum point is 1, minimum is –1.

Frequency: $b = 2$: Two complete cycles will occur within 2π.

Period: $\frac{2\pi}{2} = \pi$: π is now the interval that contains a complete cycle.

Key Points

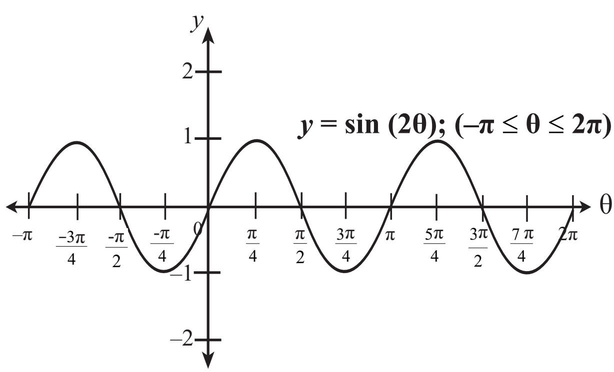
$(0, 0)$

$(\frac{\pi}{4}, 1)$

$(\frac{\pi}{2}, 0)$

$(\frac{3\pi}{4}, -1)$

$(\pi, 0)$



❸ $y = 3 \cos(\theta/2)$; $0 \le \theta \le 4\pi$
(Domain ends at 4π, label graph accordingly.)

Amplitude = 3. This will change the y values of the key points. Maximum = 3, minimum = –3

Frequency = ½ One-half of the cycle is within 2π. This will change the x values of the key points.

Period = $2\pi/(1/2) = 4\pi$. One complete cycle will occur in 4π.

Key Points: The graph requires 4π to show one cycle instead of 2π. Double the independent variable, θ, of the basic key points. The maximum and minimum y values are now 3 and –3. Multiply the basic y values of the key points by 3.

Key Points

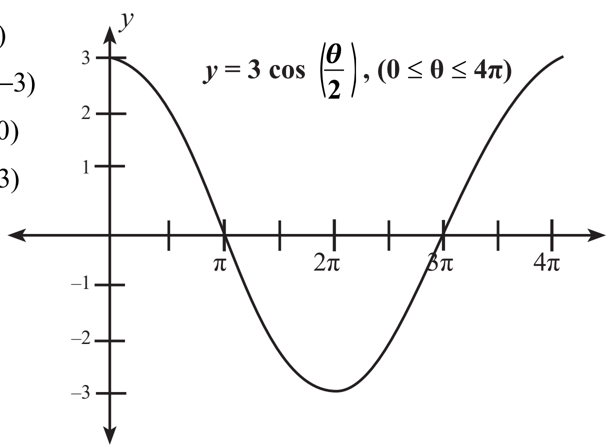
$(0, 3)$

$(\pi, 0)$

$(2\pi, -3)$

$(3\pi, 0)$

$(4\pi, 3)$



3.5

EVEN AND ODD TRIG FUNCTIONS

(See also page 87 in Unit 2.2)

Functions are described as being even, odd, or neither. Trig functions can be described in the same way. Even functions, when graphed, are symmetric to the *y*-axis. Odd functions have point symmetry with the origin. If the function has neither of these characteristics, it is described as being neither. One cycle of the trig graph demonstrates one period of the function.

Figure 1 is one cycle or one period of the sine graph. It does not appear to be odd or even. However, remember that the trig functions are cyclical.

Figure 1

$f(x) = \sin x$

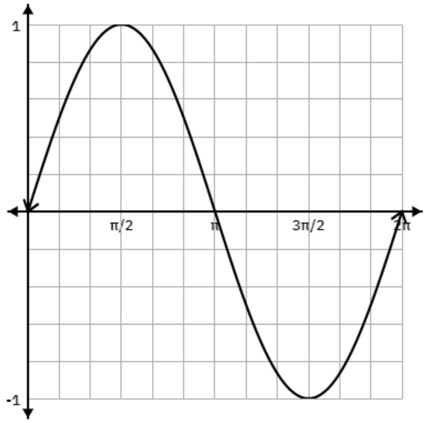


Figure 2, below shows the expanded view of the sine function and it has point symmetry with the origin. This means that if you rotate the graph around the origin, 180°, it maps to itself. $F(x) = \sin x$ is an ODD function.

Figure 2

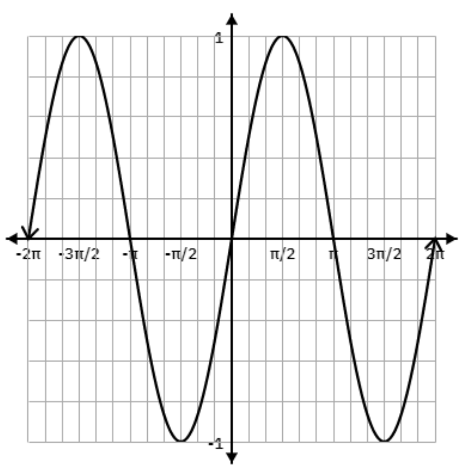
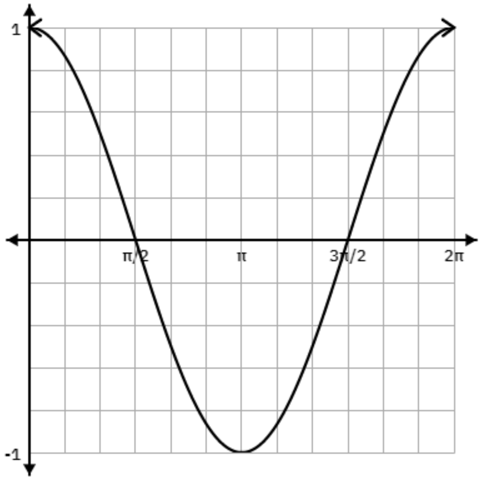


Figure 3

$f(x) = \cos x$



The graph of one cycle of the cos *x* graph in Figure 3 appears to be neither, but if it is expanded to show additional cycles of the function, it shows it is symmetric with respect to the *y*-axis. If the entire graph is reflected over the *y*-axis, it will be exactly the same. See figure 4: $F(x) = \cos x$ is an EVEN function.

Figure 4

$F(x) = \cos x$

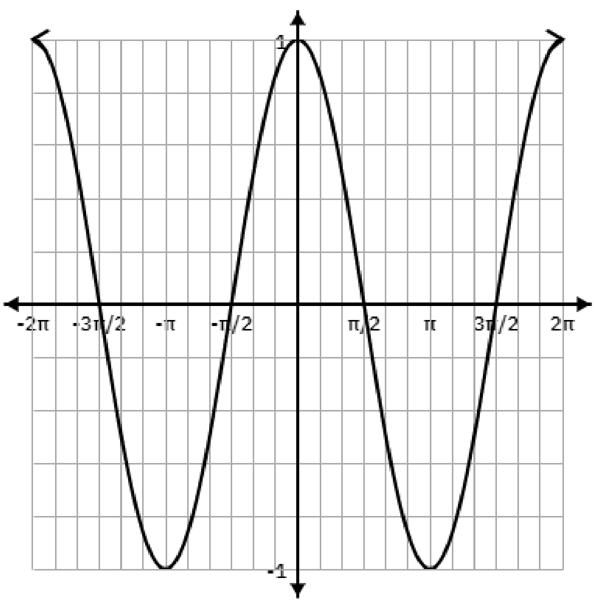
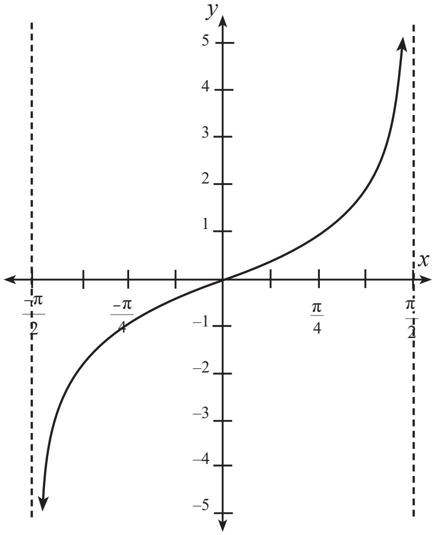


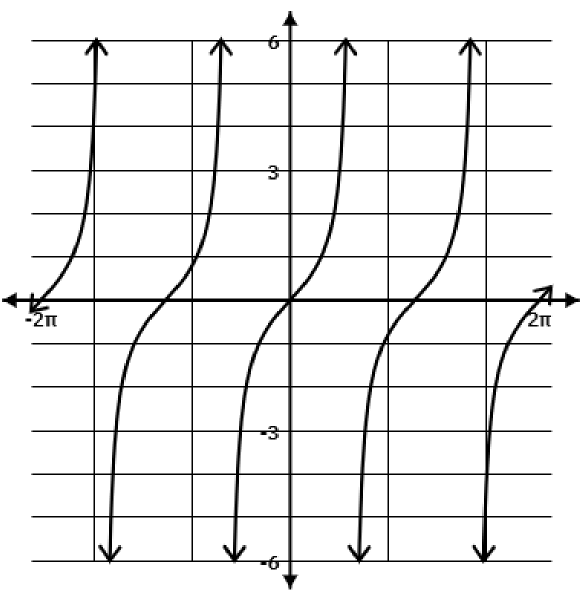
Figure 5

$f(x) = \tan x$



One cycle of the tangent graph shows that it is symmetric to the origin. See figure 5. It is an ODD function. If it is expanded to show additional cycles, the point symmetry to the origin is still evident. See figure 6.

Figure 6



Note: The period of the sine and cosine functions are each 2π and the period of the tangent function is π.

3.5

HORIZONTAL (PHASE) AND VERTICAL SHIFTS OF TRIG GRAPHS

$$y = a \sin b\,(x + c) + d$$
$$y = a \cos b\,(x + c) + d$$

In the equations shown in the previous section we used a and b to sketch the graph. (c and d were both zero.) c tells us where the graph is located horizontally on the principal axis or midline, and d tells use where the principal axis is located. (Remember that when $d = 0$, the principal axis is the x-axis.)

<u>Phase Shift</u>: A horizontal translation of the basic graph which is indicated by c.

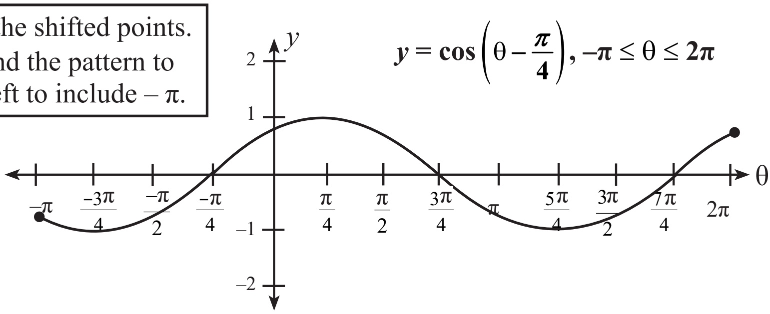
- When c is *positive*, the graph (and its key points) move $|c|$ units to the LEFT.
- When c is *negative*, the graph moves $|c|$ units to the RIGHT.

Key Points: To locate the x value of the key points, subtract c from the x (independent variable) value of the basic key points.

Example $y = \cos(\theta - \pi/4)$. Graph in the interval $-\pi \le \theta \le 2\pi$. Start with the key points of the basic graph between 0 and 2π. Since $b = 1$, subtract $-\pi/4$ from each x value. (Add $\pi/4$). Plot. Then extend repeating pattern to include the interval needed.

Key points	$\left(\text{Basic } \boldsymbol{\theta} - \left(\frac{-\pi}{4}\right), \boldsymbol{y}\right) \Rightarrow \left(\textbf{\textit{shifted}}\ \boldsymbol{\theta}', \boldsymbol{y}\right)$
$(0, 1)$	$\left(0 + \frac{\pi}{4}, 1\right) \Rightarrow \left(\frac{\pi}{4}, 1\right)$
$\left(\frac{\pi}{2}, 0\right)$	$\left(\frac{\pi}{2} + \frac{\pi}{4}, 0\right) \Rightarrow \left(\frac{3\pi}{4}, 0\right)$
$(\pi, -1)$	$\left(\pi + \frac{\pi}{4}, -1\right) \Rightarrow \left(\frac{5\pi}{4}, -1\right)$
$\left(\frac{3\pi}{2}, 0\right)$	$\left(\frac{3\pi}{2} + \frac{\pi}{4}, 0\right) \Rightarrow \left(\frac{7\pi}{4}, 0\right)$
$(2\pi, 1)$	$\left(2\pi + \frac{\pi}{4}, 1\right) \Rightarrow \left(\frac{9\pi}{4}, 1\right)$

Plot the shifted points. Extend the pattern to the left to include $-\pi$.



Vertical Shift: A vertical translation of the basic graph which is indicated by *d*.

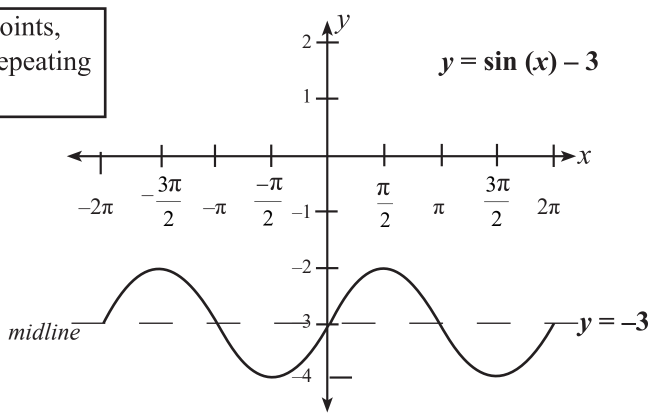
- When d is positive, the graph (and its key points) move $|d|$ units UP.
- When d is negative, the graph moves $|d|$ units DOWN.

A **Midline** is the horizontal line half way between the maximum and minimum values of a trig graph. It is the same lie as the principal axis, $y = d$.

Example $y = \sin(x) - 3$,

Key points	(Basic x, $y + d$) $\Rightarrow$ (x, *shifted* y)
(0, 0)	$(0, 0-3) \Rightarrow (0, -3)$
$(\frac{\pi}{2}, 1)$	$\left(\frac{\pi}{2}, 1-3\right) \Rightarrow \left(\frac{\pi}{2}, -2\right)$
$(\pi, 0)$	$(\pi, 0-3) \Rightarrow (\pi, -3)$
$(\frac{3\pi}{2}, -1)$	$\left(\frac{3\pi}{2}, -1-3\right) \Rightarrow \left(\frac{3\pi}{2}, -4\right)$
$(2\pi, 0)$	$(2\pi, 0-3) \Rightarrow (2\pi, -3)$

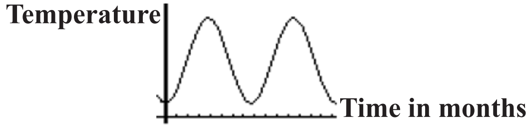
Plot the shifted points, then extend the repeating pattern to -2π.



Examples

❶ The temperatures, in Fahrenheit degrees, in a New England town, starting January 1, can be modeled by the function $f(x) = -30\cos\frac{\pi}{6}x + 40$. Graph the function on the grid below in the interval $0 \le x \le 24$ and discuss the characteristics of the graphed function. Include the midline, amplitude, maximum and minimum points, the period of the function, and any additional characteristics of interest. Write a paragraph describing these characteristics in terms of the temperature conditions in the town.

Solution: Input the function in the calculator and graph it. Use the calculations for the minimum and maximum points Determine the midline by locating the line half way between the maximum and minimum points. Then discuss the months and temperatures as they relate to the town.



Conclusion: The maximum point on the graph is (6, 70) and the minimum point is (12, 10).

The amplitude is $|a| = |-30| = 30$. The period is $\frac{2\pi}{\frac{\pi}{6}} = 12$

The midline is $y = \frac{Max\ y + Min\ y}{2} = \frac{70+10}{2} = 40$.

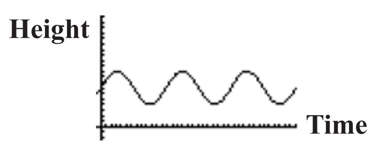
The graph is a cosine function that has been reflected over its midline, $y = 40$.

Discussion: The function begins at its minimum value of 10 which means that when the record keeping began, it was January 1 and the temperature was 10° F. The temperature increased as time continued through the months and in the 6th month, June, the maximum temperature of 70° F was reached. After that, the temperature fell until it reached its lowest point at the end of 12 months, December 31. The average temperature for the year was 40°F and the temperatures ranged from 10°F to 70°F. One complete cycle to the temperatures occurred in 12 months.

❷ In an amusement park a kiddie ride with swings operates by rotating the swings in a circle and moving them higher and lower in relation to the ground. The length of the arm from the center of the ride to the swing is 20 feet. The height of a swing, in feet, can be modeled by the function $f(t) = 3 \sin \frac{\pi}{10} t + 7$ where t represents the seconds after the ride started. Graph the function. What is the maximum height and the minimum height of the swing measured from the ground. At what height does the swing begin? How long does it take for the swing to get to its maximum height and back to its maximum height again? About how many cycles are completed in one minute?

Solution: Graph using an appropriate window in the calculator. Sketch the graph. Use the maximum and minimum values to determine the heights and the time between the maximum and minimum.

Conclusion:



The maximum height of the swing is about 10 feet and it is reached 5 seconds after the ride starts. The minimum height is about 4 feet and it is reached after 15 seconds on the ride. The swing begins at a height of 7 feet. The time for one complete cycle, from maximum height to maximum height again is 20 seconds, so three cycles will occur within a minute.

Tangent Graph:

$y = a \tan bx$ Domain $-\infty < \theta < \infty$, except odd multiples of $\frac{\pi}{2}$.

Remember that: $\tan x = \frac{\sin x}{\cos x}$. The domain of the graph is determined by this.

- When $\cos x = 0$, the tangent function is undefined. Tangent is undefined at odd multiples of $\frac{\pi}{2}$. At those values of x, a vertical asymptote indicates the line where the y values approach positive or negative infinity. The *domain* of the tangent function is $\{x\text{: R}, x \neq n \cdot \frac{\pi}{2}$ where n is an odd integer$\}$
- When $\sin x = 0$, the tangent function = 0. This happens at $\ldots, -\pi, 0, \pi, 2\pi, \ldots$

Amplitude: Tangent Graph has no amplitude, it is a vertical stretch/shrink instead. Its *range* is $-\infty < y < \infty$.

Period: π. This is a characteristic of the tangent graph. It is not determined by dividing $\frac{2\pi}{b}$ as it is in the sine and cosine graphs. Two tangent cycles appear within an interval of 2π. Use $\frac{\pi}{b}$ to find the period.

Key Points: The tangent graph has 3 points that are easy to graph and as the graph approaches the vertical asymptotes, it approaches positive or negative infinity.

Examples

❶ The key points of the basic graph are

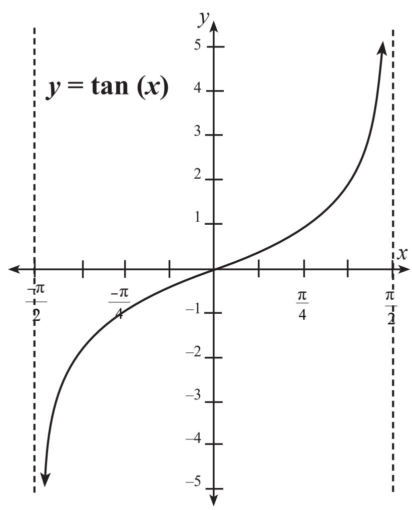
As $x \to -\frac{\pi}{2}$, $y \to -\infty$

$(-\frac{\pi}{4}, -1)$

$(0, 0)$

$(\frac{\pi}{4}, 1)$

As $x \to \frac{\pi}{2}$, $y \to \infty$



Note: An asymptote is a line that a graph approaches more and more closely.

The vertical asymptotes are the dashed lines. They repeat throughout the graph at odd multiples of $\frac{\pi}{2}$.

Shift of Trig Functions

❷ $y = \tan 2x, (-\pi \le x \le \pi)$

Asymptotes are at $\frac{-\pi}{4}$ and $\frac{\pi}{4}$.

Period on this graph $\frac{\pi}{2}$. One complete cycle occurs in an interval of $\frac{\pi}{2}$.

Key Points **$y = \tan (2x)$**

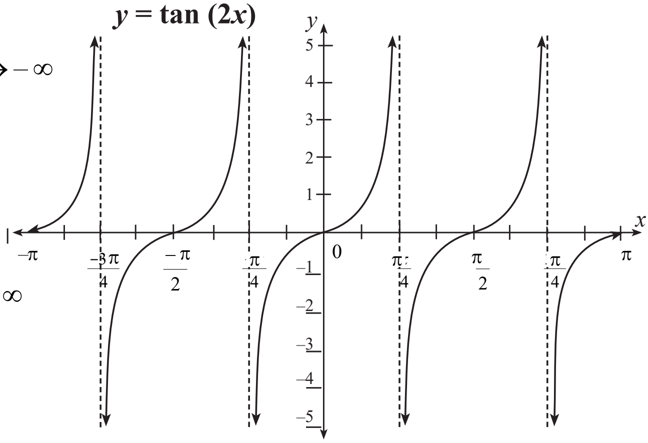
As $x \to \frac{-\pi}{4}, y \to -\infty$

$(\frac{-\pi}{8}, -1)$

$(0, 0)$

$(\frac{\pi}{8}, 1)$

As $x \to \frac{\pi}{4}, y \to \infty$



❸ $y = 3 \tan (x), (-2\pi \le x \le 2\pi)$

Asymptotes are at $x = \frac{-\pi}{2}$ and $\frac{\pi}{2}$.

Period: π

Key Points **$y = 3 \tan (x)$**

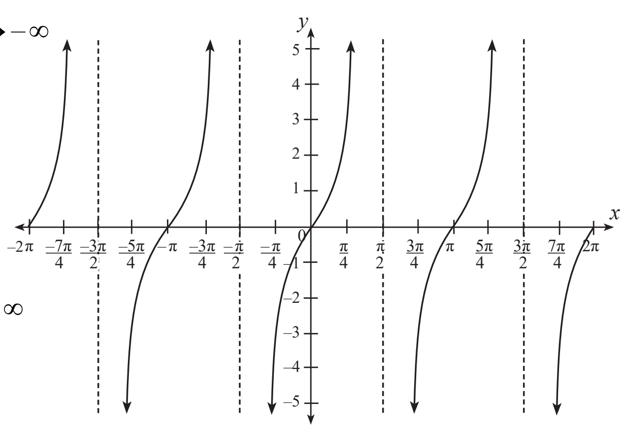
As $x \to \frac{-\pi}{2}, y = \to -\infty$

$(\frac{-\pi}{4}, -3)$

$(0, 0)$

$(\frac{\pi}{4}, 3)$

As $x \to \frac{\pi}{2}, y = \to \infty$



4.5 Displaying Bivariate Data (2 Variables)

The display of bivariate data depends on the type of data, quantitative or categorical, that is contained in the data set. Quantitative is discussed below. Categorical bivariate data can be displayed using a two-way frequency table. See page 183.

SCATTER PLOTS AND REGRESSION EQUATIONS

Quantitative Data with two variables can be displayed using a **scatter plot.** In this numerical data, the values for the ***two variables*** are paired. The plotted points often suggest a pattern (a line or a curve) which can be described using a function.

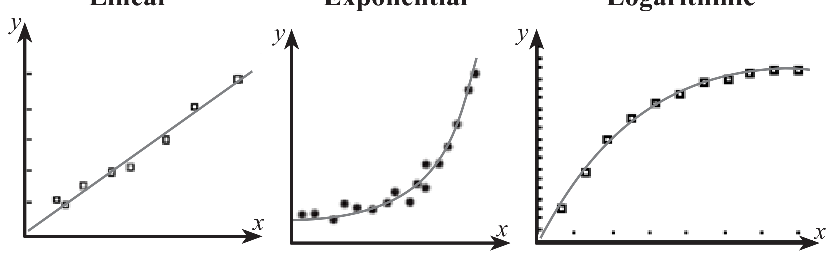
Line (or Curve) of Best Fit: It is a sketch through the points on the scatter plot such that best represents the data. A function (or equation) can be written to describe the line of best fit. This equation is also called a regression equation.

REGRESSION EQUATION

A Regression Equation is a function that represents the graph of a line or curve of best fit. Six common examples are described here but other functions such as trigonometric functions can be used. Using a calculator to develop the scatter plot as well as the regression equation provides the most accurate model. (There are more choices for regression equations depending on the data being analyzed.)

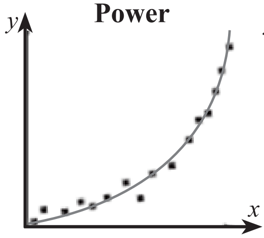
Each diagram demonstrates an example of the pattern of a scatter plot leading to a specific model for several types of regression equations. The curves of best fit are sketched in.

Linear **Exponential** **Logarithmic**

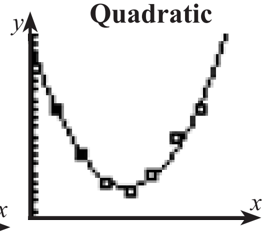


4.5

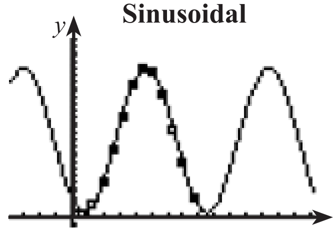
Power



Quadratic



Sinusoidal



Forms of Regression Equations

Type of Regression	Calculator Abbreviation	Form of Equation
Linear	LinReg	$y = ax + b$
Exponential	ExpReg	$y = ab^x$
Logarithmic	LnReg	$y = a + b \ln x$
Power	PwrReg	$y = ax^b$
Quadratic	QuadReg	$y = ax^2 + bx + c$
Sine	SinReg	$y = A * sin(bx + c) + d$

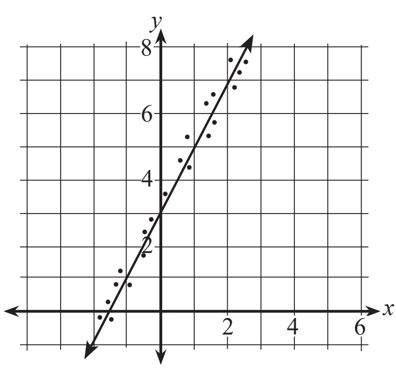
LINEAR REGRESSIONS

Correlation Coefficient, *r*: A number that indicates the strength and direction of a linear relationship. The value of *r* can be $-1 \leq r \leq 1$.

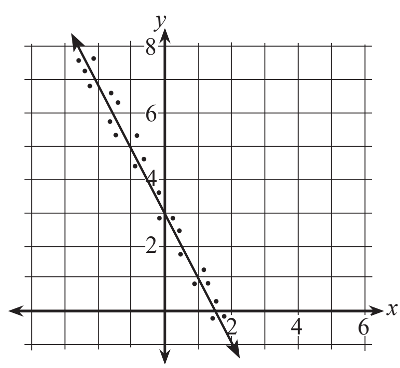
Positive or Negative Value of *r*: *Refers to the trend of the data.*

- **Positive(+)** correlation means that as x increases, y increases.
- **Negative(–)** correlation means that as x increases, y decreases.

These diagrams show a positive correlation and a negative correlation of data and a sketch of the line or curve of best fit.



Positive Correlation
$0 \leq r \leq 1$



Negative Correlation
$-1 \leq r \leq 0$

Significance of the Numerical Value of *r*: When r, the correlation coefficient, is close to +1 or –1, the correlation is "strong" meaning that the regression is a close fit to the data. The best regression for a set of data is the one that has a correlation coefficient closest to +1 or –1.

- A strong positive correlation might be $r = 0.96$. It indicates an upward trend and that the data points are very close to the line or curve of best fit.
- Strong negative: $r = -0.96$. It means a downward trend and that the data points are very close to the line of regression (or curve).
- The correlation is equally strong if $r = 0.96$ or $r = -0.96$
- As r approaches zero, from either direction, it indicates that the regression is less closely related to the data or is "weak". If $r = 0.3$ or $r = -0.3$, the correlation is about the same and it is weak. The positive r value indicates an upward pattern to the data, the negative r indicates a downward pattern.
- A zero correlation means there is no correlation between the variables. The data points are randomly scattered which makes it hard to draw a line of best fit.
- It is important to note that r is only useful when interpreting linear data.

Regression Equation: An equation that represents the line or curve of best fit.

Example Several neighbors were comparing the number of bird feeders that they have in their yards in the winter with the average amount of bird seed they used in a week. A new neighbor, James, wants to start feeding the birds using 7 feeders. How much bird seed will James use in one week?

Steps:

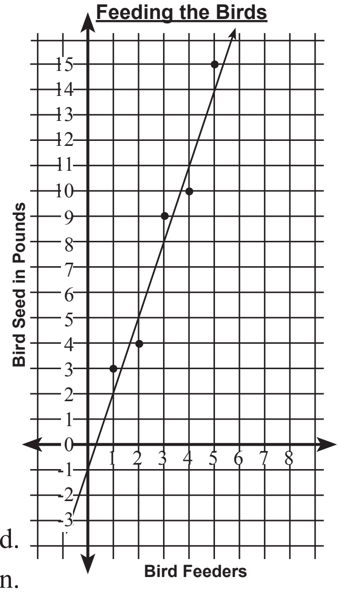
1) Create a scatter plot to demonstrate the relationship between the number of bird feeders and the pounds of bird seed used in an average week.

2) By hand, sketch a line or curve of best fit. Determine a function to define it.

3) Describe the relationship of the independent and dependent variables.

4) Create a regression equation to approximately represent the line of best fit. Substitute 7 for the value of x to find the corresponding value of y.

Bird Feeders	Bird Seed (pounds)
1	3
2	4
3	9
4	10
5	15



Solution:

a) The scatter plot for Feeding the Birds is sketched. Appropriate labels are included.

b) The data appear to follow a linear pattern. Sketch a line as accurately as possible between the data - some points maybe on it, some will not. After the line is sketched, it is necessary to define a function to describe this line.

c) When examining the data, as the independent variable (x) increases the dependent variable (y) also increases. This shows an upward or positive trend of the data.

d) The line of best fit can be sketched by hand. An approximate linear function, the regression equation in the form $y = mx + b$, can be created by reading the slope and y-intercept from the graph. Substitute 7 for x.

$m = 3$

y-intercept $= -1$

$y = 3x - 1$

$y = 3(7) - 1$

$y = 20$

e) Diagnostics must be on in the calculator. After putting the data into the calculator and creating the scatter plot, r is found using the "Stat Calc" key and choosing LinReg. The value of r appears in calculator screen. In this example $r \approx .97435$ which indicates a fairly strong positive correlation between the number of bird feeders and pounds of bird seed used. As the value of x increases, the value of y also increases. The data points are closely associated with the line of best fit.

James will need about 20 pounds of bird seed for one week. (Using the calculator instead of creating the equation using the graph, a better regression equation is $\hat{y} = 3x - .8$. In that case for 7 bird feeders, James will need about 20.2 lbs of bird seed. Very close to our approximation without the calculator in this example.) Here $\hat{y}$ (pronounced y-hat) is a symbol that represents the predicted equation in regression.

NON-LINEAR REGRESSIONS

Scatter plots from bivariate data can require a curve to approximate an appropriate regression equation instead of a line. Examine the shape of the data data points in the scatter plot and choose the regression equation that appears to be the best fit. See page 181 for diagrams illustrating non-linear regressions.

Examples

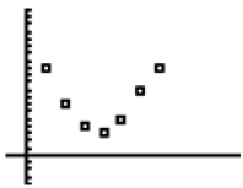
❶ An experiment results in the following data:

a) Create a scatter plot and sketch the line or curve of best fit for the data shown here.

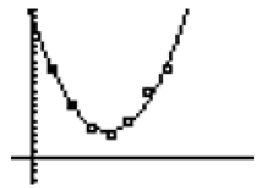
Data

X	Y
4	12
8	7
12	4
16	3
20	5
24	9
28	12

Scatter Plot



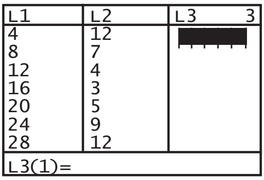
Curve of Best Fit

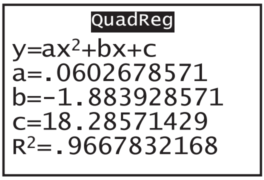


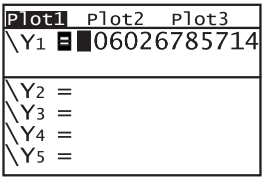
b) Use a calculator to determine the regression equation for the line or curve of best fit.

Answer: The scatter plot seems to have a shape appropriate for a quadratic function. The curve of best fit appears to be a parabola. This is called a quadratic regression, or QuadReg.

Its format will be: $y = ax^2 + bx + c$.







The equation of the curve of best fit is:

$$\hat{y} = .060267857x^2 - 1.883928571x + 18.28571429$$

c) Determine the predicted value of y when x is 2.5. (Values for y may be rounded to the nearest hundredth.)

Answer: Using the value function on the calculator, enter:

$$x = 2.5;\ \hat{y} = 13.95256696$$

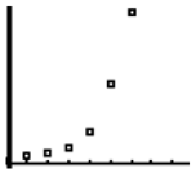
$$\hat{y} = 13.95$$

❷ Create a scatter plot for the area, in square feet, as measured on consecutive days. What type of regression equation appears to fit the data? Find the equation, rounded to the nearest thousandth, and discuss its accuracy. Using the rounded equation, what is the predicted area of square feet on the 12^{th} day, to the nearest whole number?

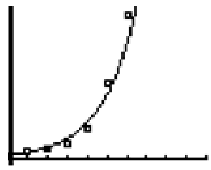
Data

Day	Area
0	2
1	4
2	5
3	7
4	14
5	35
6	66

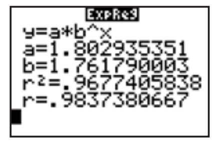
Scatter Plot



Curve of Best Fit



ExpReg



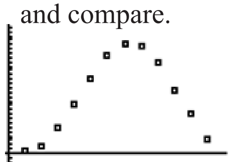
Conclusion: The scatter plot of the data and the curve of best fit appear to be related to an exponential function. The area increases exponentially as the time increases. The regression equation is $\hat{y} = 1.803(1.762^x)$. On the 12^{th} day, the area should be 1615 square feet. This value is obtained by plugging in $x = 12$.

❸ The average monthly temperatures, in Celsius degrees, in New York City over a one year period are given in the table below. January is represented by $t = 1$. Create a scatter plot and determine what type of function would best represent the curve of best fit.

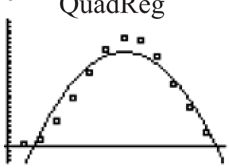
Month (t)	1	2	3	4	5	6	7	8	9	10	11	12
Temp C	0.5	1.8	5.7	11.5	16.9	22.3	25.2	24.6	20.6	14.5	9.1	3.4

Source: www.weatherbase.com

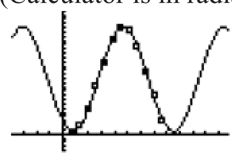
Scatter Plot: The shape of the scatter plot looks like it might be a quadratic regression or it might be a sinusoidal regression. Test both and compare.



QuadReg



SinReg (Calculator is in radian mode.)



Examining the two graphs visually, it is clear that the SinReg or Sinusoidal Regression is a much better fit for the curve of best fit than the QuadReg (quadratic regression).

Conclusion: The sinusoidal regression equation is a better fit for this data. The equation is $\hat{y} = 12.243 \sin(.515x - 2.187) + 12.827$ (Rounded to the nearest thousandth.)

4.5

PREDICTIONS USING REGRESSIONS

Substitute the value of *x* into the regression equation to find the corresponding value of *y*. If given *y*, substitute that and find *x*.

- **Interpolate:** Find a close approximate value that is within the given data
- **Extrapolate:** Find a close approximate value that is not within the given data – it is larger or smaller than the given data values. Extrapolation does not always give reliable results.

Example The data in the chart has been collected in a survey. The independent variable is in the first column and the dependent variable is in the 2nd column.

x	y
1	7
5	20
11	32
22	50
33	66
44	80
55	93

a) Determine which regression equation fits the data best – exponential, logarithmic, or power.

b) To the nearest whole number, determine the value of y when $x = 30$ and then the value of y when $x = 100$.

c) Find the value of x when $y = 100$. Round to the nearest whole number.

To solve:

a) Do all the regressions and compare the graphs to choose the best fit. Choose that equation as your answer. Show the sketch if required.

Note: Write down all the information as you work on each regression

1) **Exponential:** $a = 14.18625897$, $b = 1.040918911$
   Equation: $\hat{y} = 14.18625897(1.040918911)^x$

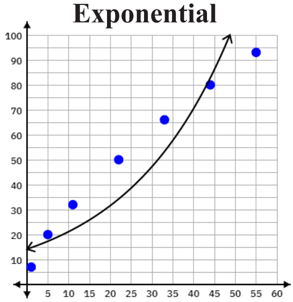
2) **Logarithmic:** $a = -5.292473504$, $b = 20.94195647$
   Equation: $\hat{y} = -5.292473504 + 20.94195647 \ln x$

3) **Power:** $a = 6.990313513$, $b = 0.6424313887$
   Equation: $\hat{y} = 6.990313513\, x^{0.6424313887}$

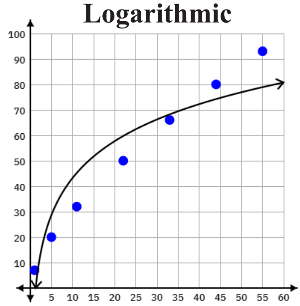
Solution: The power regression is the best fit because it follows the pattern of the data the closest.

Answer: $\hat{y} = 6.990313513\, x^{0.6424313887}$

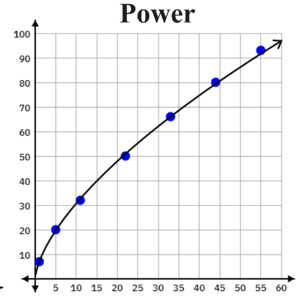
Exponential



Logarithmic



Power



b) Do the predictions. Do not round until the last step.

$x = 30$ is within the given data. This is an interpolation. Substitute.

$$\hat{y} = 6.990313513\ (30)^{0.6424313887}$$
$$\hat{y} = 62.15065544$$
$$\hat{y} \approx 62$$

$x = 100$ is outside of the given data – it is an extrapolation. Use the same method.

$$\hat{y} = 6.990313513\ (100)^{0.6424313887}$$
$$\hat{y} = 134.6974678$$
$$\hat{y} \approx 135$$

c) $y = 100$ is an extrapolation also. Now substitute for y.

$$100 = 6.990313513\boldsymbol{x}^{0.6424313887}$$

$$\frac{100}{6.990313513} = \boldsymbol{x}^{0.6424313887} \qquad \textbf{\textit{Use Logs}}$$

$$\log 100 - \log 6.990313513 = 0.6424313887 \log \boldsymbol{x}$$

$$\log \boldsymbol{x} = \frac{\log 100 - \log 6.990313513}{0.6424313887}$$

$$\log \boldsymbol{x} = 1.798640861$$

$$10^{1.798640861} = \boldsymbol{x}$$

$$\boldsymbol{x} = 62.89858286$$

$$\boldsymbol{x} \approx 63$$

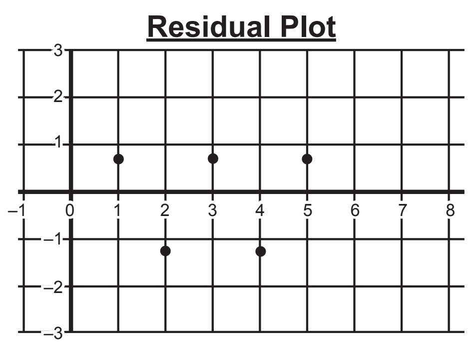
<u>Residual</u>: It is the difference between the observed y value (y_o) and the predicted y value (y_p) at each observed x value of (x_o). The residuals are another way to examine the appropriateness of the line of best fit. Ideally the residuals should be as small as possible which would indicate that the regression line was a good fit for the data. The strength of the relationship between the data and the regression function is determined by examining the location of each plotted point compared with the line or curve of best fit.

<u>Residual Plot</u>: It is a scatter plot of residual (y_r), at each observed value of x_o. The plotted ordered pairs are (x_o , y_r). A good fit would be indicated if most of the points of the residual plot are near the line $y = 0$ and have random scatter above and below the line. A pattern indicates a line is not a good fit.

Example Determine whether the line of best fit shown in the bird feeder example is a good fit for the data.

Plan: Use the regression function found using the calculator for best accuracy. Substitute the observed values of x, (x_o) in the function to find the corresponding predicted value of y_p. Find the difference between the predicted value of y_p, and the observed y_o value. We will call the difference, y_r. The calculation of y_r is: $y_o - y_p = y_r$.

Bird Feeders (x_o)	Bird Seed (lbs) (y_o)	$y_p = 3x_o - 0.8$	$y_o - y_p = y_r$	Residual Plot (x_o, y_r)
1	3	2.2	0.8	(1, 0.8)
2	4	5.2	–1.2	(2, –1.2)
3	9	8.2	0.8	(3, 0.8)
4	10	11.2	–1.2	(4, –1.2)
5	15	14.2	0.8	(5, 0.8)



Analysis: After plotting a graph of the residuals, we can see that most of the y points are close to $y = 0$. The observed points have small differences or residuals with respect to the line of best fit. The residuals are also scattered above and below the x-axis. This indicates that the line of best fit is a good fit.

Causation: Even a strong positive or negative correlation does not necessarily imply cause and effect. In the bird feeder example, the number of pounds of bird seed used does appear to be caused by how many bird feeders a person has. In other cases a strong association could be created by other variables known as lurking variables. Concluding that "*x causes y*" cannot be proved simply with the correlation coefficients and residuals.

Making Inferences and Justify Conclusions

4.5

TWO-WAY FREQUENCY TABLES

Categorical data can be presented using a two-way frequency table when there are two categorical variables involved. Within each variable there are two or more categories included. The actual count of the data is recorded in the two-way table.

Example Make a two way frequency table to record this data. Include the appropriate titles.

Data Set 1: There are 200 freshmen, 125 girls and 75 boys. 52 boys ride to school and 79 girls ride to school. The remainder of the freshmen walk.

HOW FRESHMAN GET TO SCHOOL			
Travel Method	**Gender**		**Total** (marginal row frequency)
	Boys	**Girls**	
Ride	52	79	131
Walk	23	46	69
Total (marginal column frequency)	75	125	200

What is the probability that a randomly chosen freshman is:

a) a girl?

Solution: The total number of freshmen is 200 and 125 are girls.

$$P(G) = \frac{125}{200} = .625$$

b) a walker?

Solution: Of the 200 freshman in the survey, 69 of them walk.

$$P(W) = \frac{69}{200} = .345$$

c) Is a girl who is a walker.

Solution: In the table it shows that 46 students are girls who walk.

$$P(G \text{ and } W) = \frac{46}{200} = .23$$

d) Is a girl or a walker?

Solution: Of the 125 girls, 46 of them are walkers. There are 69 walkers in all. "Or" means that the elements in both sets are included, however the 46 girls who walk cannot be counted twice. Add the probabilities and subtract the probability of the "overlap."

$$P(G \text{ or } W) = P(G) + P(W) - P(G \text{ and } W)$$

$$P(G \cup W) = P(G) + P(W) - P(G \cap W)$$

$$P(G \cup W) = \frac{125}{200} + \frac{69}{200} - \frac{46}{200} = \frac{148}{200} = .74$$

4.6

PROBABILITY

Different notations are used to describe the elements or members of a set. Sometimes the elements are listed and in other cases a rule is used to define the members within the set.

DEFINITIONS:

<u>Set</u>: A group of specific items within a universe.

Example The set of integers.

<u>Subset</u>: A set whose elements are completely contained in a larger set.

Example The set of even integers is a subset of the set of integers.

<u>Complement of a Set</u>: Symbols are A', or A^C. A' contains the elements of the universal set that are not in Set A.

Example If the universe is even numbers from 2 to 10 inclusive, and Set $A = \{2, 4, 6, 8, 10\}$, then $A' = \{3, 5, 7, 9\}$.

SYMBOLS:

$\in$ or $\subset$ means "is an element of ". $100 \in$ set of perfect squares.

$\notin$ or $\not\subset$ means "is not an element of ". $3 \notin$ set of perfect squares.

$\varnothing$ or $\{\ \}$ are symbols for the "empty set" or the "null set". The empty set or null set has no elements in it.

Example If P is the set of negative numbers that are perfect squares of real numbers, then $P = \{\ \}$ or $P = \varnothing$.

Note: Do not use $\{\ \}$ and $\varnothing$ together. $\{\varnothing\}$ means the set containing the element $\varnothing$. It does not mean the empty set or null set.

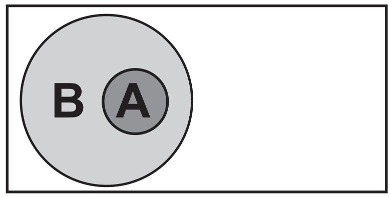
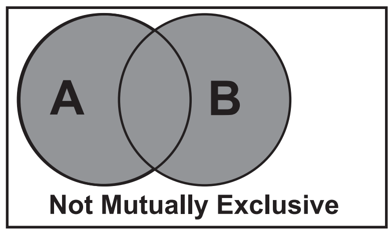
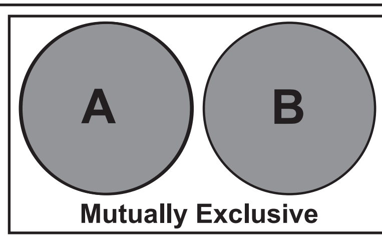
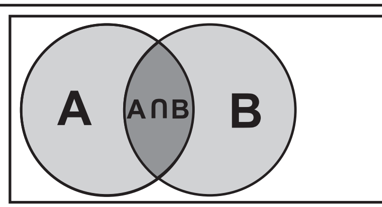
<u>Mutually Exclusive</u>: Two sets that have no elements in common are called mutually exclusive sets.

<u>Notation</u>: Ways to write the elements of a set.

4.6

VENN DIAGRAMS

Symbols are used to describe relationships in set notation. A Venn Diagram can be used to demonstrate the meaning of the symbols visually. The shaded portions of the diagrams indicate the relationship between the sets as specified by the symbol.

Meaning	Set Notation Symbol	Diagram
Subset: *A* is a subset of *B*. All the elements in *A* are also in *B*.	$A \subset B$ $A \in B$	
Union of 2 sets is the set of elements in either *A* or *B*, or in both.	$A \cup B$	
Union of 2 sets is the set of elements in both *A* and *B*.	$A \cup B$	
Intersection of 2 sets is the set of elements that are in BOTH *A* and *B*.	$A \cap B$	
Complement of a set is the elements that are in the universal set but not in the given set.	A' or A^c